Kinematic cosmology in conformally flat spacetime

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ABSTRACT

In a recent series of papers Endean examines the properties of spatially homogeneous and isotropic (FLRW) cosmological models filled with dust in the "conformally flat spacetime presentation of cosmology" (CFS cosmology). This author claims it is possible to resolve a certain number of the difficulties the standard model exhibits when confronted to observations, if the theoretical predictions are obtained in the special framework of CFS cosmology. As a by-product of his analysis Endean claims that no initial (big-bang) nor final (big-crunch) singularities occur in the closed FLRW model. In this paper we show up the fallacious arguments leading to Endean's conclusions and we consistently reject his CFS cosmology.

Subject headings: cosmology : theory — large-scale structure of universe

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1. Introduction

Kinematical aspects of Friedmann–Lemaître–Robertson–Walker (FLRW) models have been examined by Infeld & Schild (1945). These authors determined exhaustively all the possible forms of the metrics written explicitly in conformally flat form and classified all the different types of motion of free particles and light rays in these various universes. In their study they willfully ignored the actual dynamics of the cosmological models. This very problem was tackled by Tauber (1967) who solved explicitly Einstein's equations for the FLRW conformally flat form metrics and for various types of equation of state.

The conformally flat spacetime presentation of cosmology (CFS cosmology) has been developed in a recent series of papers. It differs from Tauber's approach since it takes root in Infeld & Schild's kinematic cosmology and does not analyze the dynamical behaviour of the conformal factor. Endean (1994) examines the consequences of expressing redshift-distance relations and the timing test in terms of the CFS coordinates for the open FLRW model filled with dust. Endean (1995) extends his previous analysis to the closed FLRW dust model taking into account partial determinations of the Hubble constant and statistical analysis of redshift-distance data at intermediate range. He claims that CFS cosmology resolves the cosmological age and redshift-distance difficulties exhibited by these observational data. Endean (1997) asserts that CFS time provides the true measure of elapsed time in the universe and that no spacetime singularities occur in the closed FLRW dust model.

In this paper we briefly summarize the key results of kinematic cosmology along the lines of Infeld & Schild (1945). We point out the interpretational difference between this framework and Endean's CFS cosmology. On this basis we show up the fallacious arguments underpinning Endean's conclusions, and accordingly we reject Endean's spurious cosmology.

2. Kinematic cosmology

2.1. Equivalence of FLRW and conformally flat metrics

Kinematic cosmology rests on the property that FLRW models are conformally flat, i.e. related to some regions of flat spacetimes by means of a conformal transformation of the metric, viz. $\widetilde{g}_{ab} = \varphi^2(x) g_{ab}$ where φ is the conformal factor. Without performing such transformation it is always possible to find a transformation of coordinates that brings the initial FLRW metrics to manifestly conformally flat forms. One starts from the angular form of FLRW metrics in comoving coordinates (τ, ρ),

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + R^2(\tau)\left[\mathrm{d}\rho^2 + \chi^2(\rho)\mathrm{d}\omega^2\right], \qquad (1)$$

where R denotes the scale factor of the universe, τ is the proper cosmic time, ρ is the radial comoving coordinate, ω stands for the spherical solid angle, and the functions $\chi(\rho)$ read explicitly for the closed $(k = +1)$, flat $(k = 0)$ and open $(k = -1)$ models respectively as $\sin\rho$, ρ and $\sinh\rho$. Upon introducing the 'development angle' η as a new time coordinate, $\mathrm{d}\tau = \pm R(\eta)\mathrm{d}\eta$, and null coordinates (ξ, λ) by $\xi = \frac{1}{2}(\eta + \rho)$, and $\lambda = \frac{1}{2}(\eta - \rho)$, the metric (1) takes the form

$$\mathrm{d}s^2 = R^2(\xi + \lambda)\left[-4\mathrm{d}\xi\mathrm{d}\lambda + \chi^2(\xi - \lambda)\mathrm{d}\omega^2\right]. \qquad (2)$$

The transformation that turns the metric (2) into a conformally flat metric is defined by $X = g(\xi) = t + r$ and $Y = g(\lambda) = t - r$, where (t, r) are the CFS coordinates, and the function $g(w)$ is given explicitly for the closed, flat and open models respectively by $\tan w$, w and $\tanh w$ (Lightman et al. 1975). With this transformation the metric (2) becomes

$$\mathrm{d}s^2 = \gamma(r, t)\left[-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\omega^2\right], \qquad (3)$$

where the conformal factor satisfies the identity

$$\gamma(r, t) = \frac{4R^2\left[g^{-1}(t + r) + g^{-1}(t - r)\right]}{g'(t + r)g'(t - r)}, \qquad (4)$$

g' and g^{-1} denoting respectively the derivative of $g(w)$ and the inverse function to $g(w)$. The relationship between the comoving coordinates (η, ρ) and the CFS coordinates is readily obtained from the above definitions. It reads

$$t \pm r = g\left(\frac{1}{2}(\eta \pm \rho)\right). \qquad (5)$$

At this stage it is important to note that if one takes up the CFS metric (3) instead of the conventional metric (1) one shifts the study of the space structure, i.e. the knowledge of the scale factor, which characterizes the dynamics of the model under consideration to the analysis of the types of motion of fundamental

particles (kinematics). In total agreement with Infeld & Schild (1945) we stress here that these types of motions have no straightforward physical interpretation. If one's aim is to study the dynamics in CFS coordinates one has to follow Tauber's (1967) approach, i.e. solve Einstein's equations for the conformal factor $\gamma(r,t)$. The would-be advantage of choosing the CFS metric (3) lies in the fact that light rays propagate on straight lines with constant velocity in CFS coordinates (because conformal transformations preserve the light cone structure — of Minkowski spacetime in this case). In the line of Infeld & Schild (1947) one can set a good illustration of this property by deriving a general formula for the redshift of distant objects in CFS coordinates.

2.2. General formula for the redshift

Consider an atom, moving with a fundamental particle $P = (r,t)$ of radial velocity $v(r,t) = \mathrm{d}r/\mathrm{d}t$ and emitting photons with a proper period $\mathrm{d}\tau$ in the direction of an observer at $(0,t_0)$, where t_0 is the present CFS time. The corresponding null geodesics of the CFS metric (3) are straight lines, $t_0 = t + r$. The proper emitted wavelength is $\lambda_\mathrm{e} = \mathrm{d}\tau = \gamma^{1/2}\sqrt{1-v^2}\,\mathrm{d}t$, and the proper observed wavelength is $\lambda_0 = \mathrm{d}\tau_0 = \gamma^{1/2}(0,t_0)(1+v)\mathrm{d}t$. Therefore the redshift is given by

$$1+z = \frac{\lambda_0}{\lambda_\mathrm{e}} = \left(\frac{\gamma_0}{\gamma}\right)^{1/2}\left(\frac{1+v}{1-v}\right)^{1/2}. \qquad (6)$$

One clearly sees that in CFS coordinates there are two contributions to the redshift, namely the gravitational and Doppler effects. Note that equation (6) could be obtained by performing the transformation (5) on the standard formula

$$1+z = \frac{R(\eta_0)}{R(\eta)}. \qquad (7)$$

At this stage we arrive at the very key-stone of Endean's work since this author looks at the consequences of expressing the redshift formula (7) in terms of the CFS coordinates. There are no a priori objections against such a procedure provided one bears in mind the unphysical nature of CFS coordinates. However, Endean overrules the fundamental postulate of general relativity, namely the general covariance of the physical laws, by ascribing an absolute physical meaning to CFS coordinates. It should be clear that this infringment is simply not acceptable for there is no reason whatsoever to forsake the coordinate invariance of physical laws. For instance, Endean (1997) claims that CFS time t correctly measures the true age of the universe whereas, by definition, the genuine elapsed time is the proper cosmic time τ. Moreover, in contrast to Infeld & Schild (1945), Endean merges purely kinematic concepts into dynamical ones, asserting that the radial velocity $v(r,t)$ may be interpreted as the cosmological recession velocity. Underpinning his arguments on such mistaken assumptions conveys Endean to startling conclusions as for instance the vanishing of spacetime singularities (see §3.3.).

3. CFS cosmology

For the sake of briefness we now restrict ourself to the closed FLRW cosmological model with dust in order to show up the fallacies present in Endean (1997). For our purpose we focus the discussion on the behaviour of the scale factor and of the cosmological parameters expressed in CFS coordinates.

3.1. Scale factor and t-clock

The exact solution of Einstein's equations describing a closed FLRW model filled with dust can be found in any textbook on cosmology and reads, in terms of η,

$$R(\eta) = K(1-\cos\eta), \qquad (8a)$$
$$\tau = K(\eta - \sin\eta), \qquad (8b)$$

where K is a constant of integration which could be determined in principle from the knowledge of the present values of the Hubble constant H_0 and the mass density of the universe, $K = \Omega_0/2H_0\,(\Omega_0-1)^{3/2}$ with Ω_0 denoting the present density parameter. On the other hand, the time flow in CFS coordinates is not the same as the proper cosmic time flow. The relationship between a τ-clock and a t-clock is indeed given by

$$\mathrm{d}\tau_0 = \gamma^{1/2}(0,t_0)\mathrm{d}t_0 = K\sin^2\eta_0\,\mathrm{d}t_0, \qquad (9a)$$
$$t_0 = \tan\left(\frac{\eta_0}{2}\right). \qquad (9b)$$

Turning aside from the interpretation of equations (9) Endean (1997) states a new "principle of identical clock rates" asserting that the flows of CFS and cosmic times are identical, viz. $\mathrm{d}\tau_0 \equiv \mathrm{d}t_0$. This, a

major offense, shows unambiguously that CFS cosmology is fundamentally unsound. For instance, this "principle" has the awkward consequence that K is no longer constant but determined as a function of η, namely $K(\eta) = 1/\sin^2\eta$. This is in total contradiction with the exact solution (8). In that case the (wrong) scale factor becomes $R(\eta) = 1/(1+\cos\eta)$ and tends to infinity as η tends to π. On account of these misconceptions Endean (1997) deduces that the scale factor can not reach its maximum value either in a finite CFS time or in a finite cosmic time, and that this maximum is infinite. As another consequence of this "principle" Endean finds that the conformal factor must fulfill the condition $\gamma(0, t_0) = 1$ for all t_0. He then infers from this result a property he calls a "strong Copernician principle" enabling him to conclude that CFS coordinates measure the true distances and elapsed times in the universe and that these measures do not depend on the location of the observers. The "strong Copernician principle" is obviously wrong, for the conformal factor (4) becomes in the closed FLRW model,

$$\begin{aligned}\gamma(r,t) &= 4K^2\left[\sqrt{1+X^2}\sqrt{1+Y^2}-(1-XY)\right]^2/ \\ &\quad \left[1+X^2\right]^2\left[1+Y^2\right]^2, \end{aligned} \tag{10}$$

where X, Y have been defined above and $\gamma(r,t) = 1$ only for $r = 0$ and $t = 1$.

3.2. Cosmological parameters

From the exact solution (8) it is straightforward to obtain the Hubble and deceleration parameters as functions of η. They read respectively

$$H(\eta) := \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}\tau} = \frac{\sin\eta}{K(1-\cos\eta)^2}, \tag{11a}$$

$$q(\eta) := -\frac{1}{RH^2}\frac{\mathrm{d}^2R}{\mathrm{d}\tau^2} = \frac{1}{2}\left[1+\tan^2\left(\frac{\eta}{2}\right)\right] \tag{11b}$$

Taking equation (9b) into account their present values in terms of a t-clock are readily obtained, $H_0 = (1+t_0^2)/(2Kt_0^3)$, and $q_0 = (1+t_0^2)/2$. A usual redshift-distance relation is written down upon expanding the redshift formula (7) around small proper distances l,

$$z \sim H_0 l + \frac{1}{2}(1+q_0)H_0^2 l^2 + \mathcal{O}(l^3). \tag{12}$$

If the expansion is performed around small CFS radial coordinates r, one obtains a similar expression, viz.

$$z \sim \frac{2}{t_0\left(1+t_0^2\right)}r + \frac{5t_0^2+3}{t_0^2\left(1+t_0^2\right)^2}r^2 + \mathcal{O}(r^3). \tag{13}$$

In analogy with equation (12) Endean (1994) identifies the coefficient in front of the first order term in equation (13) as a new Hubble parameter $\bar{H}_0$ which would be the observed Hubble parameter if CFS coordinates were physically meaningful. As we know that it is not the case, the introduction of $\bar{H}_0$ is completely irrelevant. (Notice also that the "principle of identical clock rates" implies $\bar{H}_0 = H_0$.) Endean's (1997) subsequent discussion on the timing test in CFS coordinates is therefore unnecessary.

3.3. Spacetime singularities

The occurrence of spacetime singularities is a generic feature of spatially homogeneous models (Hawking & Ellis 1973). All FLRW cosmological models exhibit a singularity in the past (big-bang) where at $\tau = 0$ the mass-energy density goes to infinity. In addition, the closed model recollapses towards a singularity in the future (big-crunch) after a finite proper time. In total contradiction with these facts, Endean (1997) concludes that neither the big-bang nor the big-crunch take place in the closed FLRW model if it is examined from the point of view of CFS coordinates. With regard to the initial singularity he grounds his reasoning on the fact that the radial CFS velocity $v(r,t)$ (mistakenly considered as the recession velocity), vanishes when $t \to 0$. We prove now, if necessary, that the initial singularity fairly exists in CFS coordinates. For our purpose we consider the curvature invariant constructed on the Riemann tensor, viz. $I = R_{klmn}R^{klmn}$. Its expansion as the CFS time tends to zero is given at the leading order by

$$I \sim \frac{15}{16K^4}\left(1+r^2\right)^{12}t^{-12} + \mathcal{O}\left(t^{-10}\right), \tag{14}$$

and thus diverges. Note that Tauber (1967) also found a diverging energy density. Concerning the final singularity now, it is obvious from equation (9b) that if one ascribes a physical meaning to a t-clock as in Endean (1997) then the universe will never be seen to recollapse since $t_0 \to \infty$ as $\eta_0 \to \pi$, the value $\eta_0 = \pi$ corresponding to the maximum radius (see eq. [8]). Therefore, Endean's claim of a non-recollapsing closed universe is also totally wrong.

4. Discussion

In this paper we have shown up the fallacious assumptions founding Endean's CFS cosmology. Our conclusions can be summarized as follows. First of

all, CFS coordinates do not measure true distances and elapsed times in the universe. This is an obvious statement on account of the definition of CFS coordinates and of the mere inspection of the CFS metric (3). (Coordinates are nothing else than labels for events occurring in spacetime, and it is the metric, eventually, that tells us how to measure distances and time intervals.) On the one hand Endean's opposite claim with respect to elapsed times rests on his totally unjustified "principle of identical clock rates." On the other hand his assertion stating that the CFS radial distance represents a true measure of distance is partly based on a direct consequence of this "principle," and partly justified from a suitable fitting of redshift-distance curves (Endean 1995). We stress however that reproducing curve shapes can never replace a reliable theoretical framework. Secondly, the occurrence of singularities in FLRW cosmological models is by no means altered upon introducing CFS coordinates — Physical results are coordinate invariant. Endean's opposite conclusion reveals that his argumentation rests on the serious misconceptions mentioned above concerning the absolute physical meaning ascribed to CFS coordinates which breaks the general covariance principle. On account of these facts we conclude that Endean's CFS cosmology is totally unsound and ought to be discarded.

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